INVARIANT SUBRINGS OF SEPARABLE ALGEBRAS

BY

ANDY R. MAGID

ABSTRACT

This paper gives a necessary and sufficient condition that the ring of invariants of every group of automorphisms of every projective, separable, commutative algebra over a given commutative ring is itself a union of separable, projective subalgebras. Rings satisfying the condition include products of connected rings, von Neumann regular rings, and some rings of functions.

Let R be a commutative ring and S a commutative, separable R-algebra, finitely generated and projective as an R-module (that is, S is a *strongly separable* R-algebra). Let G be a group of R-algebra automorphisms of S, and consider the ring S^G of elements of S invariant under G. This paper studies the separability properties that S^G inherits from S.

If G is finite, by [4, (0.9), p. 709], S^G is a separable R-algebra, and if R has only finitely many idempotents then by [9, (1.3), p. 723] G must be finite. Thus we are mainly concerned with the case where R has infinitely many idempotents, and we are looking for a necessary and sufficient condition on R such that for every S and G as above, S^G is a direct limit of strongly separable subalgebras, that is, a *locally strongly separable* algebra.

The condition, stated in Theorem 22, is essentially that the Pierce sheaf [7, (4.4), p. 17] of every strongly separable R-algebra is a Hausdorff topological space. We do not deal explicitly with the Pierce sheaf here, however, but with the purely algebraic description of it given in [10]; the forementioned condition will also be phrased purely algebraically.

We use the conventions of [5] throughout: all rings and algebras are commutative, with R the generic base ring. $X(R)$ is the space of connected components

Received May 18, 1973

of Spec (R); elements of $X(R)$ are maximal ideals in the Boolean ring of idempotents of R and the basic open-closed sets of $X(R)$ are the sets $N(e) = \{x \in X(R) : 1-e \in x\}$, where e is an idempotent of R. If $x \in X(R)$ and M is an R-module, $M_x = M/Mx$, and if $m \in M$, m_x is the image of m in M_x.

DEFINITION 1. Let $a \in R$, and let $Z(a) = \{x \in X(R) : a_x = 0\}$. If $Z(a)$ is a closed subset of $X(R)$, a is *zero-closed*. If every a in R is zero-closed, R is *zero-closed*.

By [10, (2.9), p. 87], $Z(a)$ is open for all a in R. Thus a is zero-closed if and only if $Z(a) = N(e)$ for some idempotent e of R.

In the language of [7], R is zero-closed if and only if the Pierce sheaf on $X(R)$ is Hausdorff.

LEMMA 2. *Let $b \in R$. Then b is annihilated by no non-zero idempotent if and only if $Z(b)$ is empty.*

PROOF. If $eb = 0$ with $e \neq 0$, there is $x \in X(R)$ with $e_x = 1$, so $0 = (eb)_x = b_x$ and $x \in Z(b)$. Conversely, if $x \in Z(b)$, by [10] there is e in x with $e_x = 1$ such that $eb = 0$.

PROPOSITION 3. *An element a of R is zero-closed if and only if $a = eb$ where e is an idempotent of R and b is annihilated by no non-zero idempotent.*

PROOF. Let a be zero-closed, say $Z(a) = N(e)$. Then $a_x = 0$ if and only if $e_x = 1$. Let $b = a + e$ and let $x \in X(R)$. If $x \in Z(a)$, $b_x = e_x = 1$ and if $x \notin Z(a)$, $b_x = a_x \neq 0$. So $Z(b)$ is empty and, by Lemma 2, b is annihilated by no non-zero idempotent. Finally, we have $(1-e)b = a$: for $(1-e)b = (1-e)a$, and if $x \in Z(a)$, $((1-e)a)_x = 0 = a_x$ and if $x \notin Z(a)$, $((1-e)a)_x = a_x$, so for all $x \in X(R)$, $((1-e))a_x = a_x$. By [10, (2.9), p. 87], this means $(1-e)a = a$.

Now suppose $a = eb$ as in the statement of the proposition. Since $e_y = 0$ implies $a_y = 0$, $N(1-e) \subset Z(a)$. If $a_y = 0$ and $e_y = 1$, $b_y = 0$ so $y \in Z(b)$. Since $Z(b)$ is empty, $a_y = 0$ implies $e_y = 0$, and hence $Z(a) = N(1-e)$ is closed.

We record some easy consequences of the proposition.

COROLLARY 4. *Let R be zero-closed and let T be a subring of R containing all idempotents of R. Then T is zero-closed.*

PROOF. Let $a \in T$. As an element of R, $a = eb$ where b is annihilated by no non-zero idempotent and e is an idempotent. By hypothesis, $e \in T$, and by the proof of the proposition we may assume $b = a + f$, f idempotent, so $b \in T$. The proposition now implies that a is zero-closed.

COROLLARY 5. *Let $\{R_i\}$ be a collection of zero-closed rings and let $R = \Pi R_i$. Then R is also zero-closed.*

PROOF. Let $a = (a_i)$ be in R. Write, using the proposition, $a_i = e_i b_i$, where e_i, b_i are as in the proposition. Then $e = (e_i)$ is an idempotent of R while $b = (b_i)$ is annihilated by no non-zero idempotent of R. Thus $a = eb$ is zero-closed by the proposition.

COROLLARY 6. *If R is a von Neumann regular ring* [7, p. 40], *R is zero-closed.*

PROOF. Every element of R is an idempotent times a unit. (This corollary also follows from [7, (10.5), p. 43].)

If R is connected (that is, has no non-trivial idempotents), $X(R)$ is a single point, so R is zero-closed. By Corollary 5, any product of connected rings is zero-closed. The next result adds to the collection of examples.

LEMMA 7. *Let R_0 be a topological ring whose only idempotents are zero and one. Let X be a topological space and let $R = C(X, R_0)$ be the ring of all R_0-valued functions on X. Then $f \in R$ is zero-closed if and only if $f^{-1}(0)$ has a maximal open-closed subset.*

PROOF. First, let $b \in R$. If $b^{-1}(0)$ contains an open-closed subset U and e is the idempotent characteristic function of U then $eb = 0$, and if U is non-empty, $e \neq 0$. Conversely, if e is a non-zero idempotent and $eb = 0$, $e^{-1}(1)$ is a non-empty open-closed subset of $b^{-1}(0)$. Thus b is annihilated by no non-zero idempotent if and only if $b^{-1}(0)$ contains no non-empty open-closed subsets. Now if $f \in R$ is zero-closed, by Proposition 3 $f = eb$ where e is idempotent and $b^{-1}(0)$ has no non-empty open-closed subsets. Then $f^{-1}(0) = e^{-1}(0) \cup b^{-1}(0)$ has $e^{-1}(0)$ as a maximal open-closed subset. Conversely, suppose $f^{-1}(0)$ has U as maximal open-closed subset and let e be the characteristic function of U. Let $b = f + e$. Then $b^{-1}(0) \subset f^{-1}(0) - U$ has no open-closed subsets. Also, we have $(1 - e)b = f$, and by the first part of the proof and Proposition 3, f is zero-closed.

COROLLARY 8. *If R_0 is a discrete topological ring with no idempotents except zero and one, and X is totally disconnected compact Hausdorff space, $C(X, R_0)$ is zero-closed.*

PROOF. In this case all zero sets in X are open-closed.

COROLLARY 9. *A weakly uniform ring* [2, (2.3), p. 305] *is zero-closed.*

PROOF. A weakly uniform ring is a finite product of rings of the type of Corollary 8, and Corollary 5 gives the result.

COROLLARY 10. *Let X be a totally disconnected compact Hausdorff topological space and R_0 the reals or complexes. Then $C(X, R_0)$ is zero-closed if and only if X is basically disconnected* [3, (1H), p. 22].

PROOF. A closed subset Y of X has a maximal open-closed subset if and only if the interior of Y is open-closed. The interiors of zero sets are closed if and only if the closure of cozero sets are open, and hence the result follows.

It will be necessary to know in the subsequent discussion when a strongly separable algebra over a zero-closed ring is again zero-closed. Corollary 10 will be used to show that some hypothesis on the base is necessary.

Let X be a totally disconnected compact Hausdorff space and let g be a continuous real-valued function on X such that $g^{-1}(0)$ is non-empty with empty interior. Let Y be the union of two copies X_1 and X_2 of X with the images of $g^{-1}(0)$ in X_1 and X_2 identified. Y is also a compact, totally disconnected Hausdorff space. Let f be the continuous real-valued function on Y which agrees with g on X_1 and is zero on X_2. Then the interior of $f^{-1}(0)$ is $X_2 - g^{-1}(0)$ which is not closed in Y. Specializing X and g properly will now give the desired example.

EXAMPLE 11. Let X be the Stone-Čech compactification of the discrete space of positive integers, and let $g \in C(X, \mathbb{R})$ be $g(u) = 1/u$ for u a positive integer. Then $g^{-1}(0) = F$ is non-empty but has empty interior. Let Y be the space constructed as above. Let the cyclic group of order 2 act on $C(Y, \mathbb{C})$ as follows: the non-identity automorphism σ acting on the function h is to be $\sigma(h)(y) = \overline{h(y^*)}$ where y^* is the element of Y in the other copy of X from y (or y itself if $y \in F$) and $(^-)$ denotes complex conjugation. The usual arguments show that the ring of invariants of $C(Y, \mathbb{C})$ under this action is $R = \{f \in C(X, \mathbb{C}) : f(x) \in \mathbb{R} \text{ for all } x \in F\}$ and that $C(Y, \mathbb{C})$ is a Galois [1, (1.4), p. 20] extension of R. Since X is extremely disconnected [3, (6u), p. 96], it is basically disconnected [3, (1H), p. 22] and $C(X, \mathbb{C})$ is, by Corollary 10, zero-closed. By Corollary 4, R is zero-closed. But, by Corollary 10 and the above discussion, $C(Y, \mathbb{C})$ is not zero-closed.

Because of Example 11, we make the following definition.

DEFINITION 12. R is *extensionally zero-closed* if every strongly separable R-algebra is zero-closed.

PROPOSITION 13. *The following rings are extensionally zero-closed*:

(i) *Connected rings,*

(ii) *von Neumann regular rings, and*

(iii) *Weakly uniform rings.*

PROOF. (i) A strongly separable algebra over a connected ring is a finite product of connected rings, hence zero-closed.

(ii) A strongly separable algebra over a von Neumann ring is again a von Neumann ring, and hence zero-closed.

(iii) A strongly separable algebra over a weakly uniform ring is again weakly uniform [6, (2.16), p. 118] and hence zero-closed by Corollary 9.

PROPOSITION 14. *Let $\{R_i\}$ be a collection of extensionally zero-closed rings. Then $R = \Pi R_i$ is extensionally zero-closed.*

PROOF. Let S be a strongly separable R-algebra. Since S is a finitely generated projective R-module, $S = \Pi(S \otimes_R R_i)$. Each $S \otimes_R R_i$ is strongly separable over R_i and hence zero-closed by hypothesis. Then S is zero-closed by Corollary 5.

The connection between extensionally zero-closed rings and fixed rings of groups is given in the next result.

PROPOSITION 15. *Suppose R is extensionally zero-closed. Let S be a strongly separable R-algebra and let G be a group of R-algebra automorphisms of S. Then S^G is a locally strongly separable R-algebra.*

PROOF. First, it will be shown that S can be taken to be weakly Galois over R in the sense of [10, (3.1), p. 90], for there is, by [8, p. 166], a weakly Galois R-algebra T with $R \subset S \subset T$. By [10, (3.12), p. 94], T is weakly Galois over S. Since T is a finitely generated projective R-module, $(S \otimes_R T)^G = S^G \otimes_R T$, and hence we may replace R by T and S by $S \otimes_R T$ so that S is weakly Galois (note that T is still extensionally zero-closed). The remainder of the argument is analogous to that of [5, (3.12), p. 102]: for each $\sigma \in G$, there is a homomorphism $S \otimes_R S \to S$ by $s \otimes t \to s\sigma(t)$. Let $\mathrm{Tr}(\sigma)$ be the image of $X(S)$ in $X(S \otimes_R S)$ under the induced continuous map. Since the kernel of the homomorphism is idempotent generated [6, (2.4), p. 114], an element a of $S \otimes_R S$ is in the kernel if and only if $\mathrm{Tr}(\sigma) \subset Z(a)$. Thus s in S is in S^G if and only if $\mathrm{Tr}(\sigma) \subset Z(s \otimes 1 - 1 \otimes s)$ for all $\sigma \in G$. Let H be the intersection of all open-closed subgroupoids of $X(S \otimes_R S)$ [5, (1.8), p. 93] which contain $\mathrm{Tr}(\sigma)$ for all $\sigma \in G$. Since $S \otimes_R S$ is strongly separable over R, $s \otimes 1 - 1 \otimes s$ is zero-closed for all $s \in S$, and hence $Z(s \otimes 1 - 1 \otimes s)$ is an open-closed subgroupoid of $X(S \otimes_R S)$ for all $s \in S$. Thus $s \in S^G$ if and only if $H \subset Z(s \otimes 1 - 1 \otimes s)$. Then by [5, (1.10), p. 95], $S^G = S^H$ is locally strongly separable over R.

There is a weaker version of Proposition 15, which we state as a corollary.

COROLLARY 16. *Suppose R is zero-closed. Let S be a strongly separable R-algebra generated over R by idempotents, and let G be a group of R-algebra automorphisms of S. Then S^G is a locally strongly separable R-algebra.*

PROOF. For each x in $X(R)$, S_x and $(S \otimes_R S)_x$ are isomorphic to a finite product of copies of R_x. It is easy to see that S is a weakly Galois R-algebra. The argument of Proposition 15 will work in this case, once the following lemma is established.

LEMMA 17. *Let R be zero-closed and let T be an R-algebra such that for each x in $X(R)$, T_x is a finite product of copies of R_x. Then T is zero-closed.*

PROOF. Let $s \in S$, $y \in X(S)$, and suppose $s_y \neq 0$. Using the usual Boolean spectrum techniques on S_x where $x = y \cap R$, we can write $s = \Sigma a_i e_i$ where $e_i e_j = 0$ if $i \neq j$, the e_i are idempotents of S, the a_i are in R, and $(a_1 e_1)_y \neq 0$. By Proposition 3, $a_1 = fb$ where f is an idempotent of R and $b_x \neq 0$ for all x in $X(R)$. Then $b_z \neq 0$ for all $z \in X(S)$ also, and $f_y \neq 0$ since $(a_1)_y \neq 0$. Moreover, $e_1 f \neq 0$ since $a_1 e_1 = e_1 fb \neq 0$. Now let $w \in N(fe_1)$. Then $s_w = (a_1 e_1)_w = b_w \neq 0$, and $y \in N(e_1 f) \subset X(S) - Z(s)$. Thus $Z(s)$ is closed.

The next step is to establish the converses of Corollary 16 and Proposition 15. This requires some additional notation.

DEFINITION 18. Let U be a subset of $X(R)$. Then $T(U)$ denotes the R-subalgebra of $R \times R$ generated by all pairs $(0, a)$ such that $U \subset Z(a)$.

LEMMA 19. *Let U be an open subset of $X(R)$. Let G be the group of all R-algebra automorphisms of $R \times R$ fixing $T(U)$. Then $T(U) = (R \times R)^G$.*

PROOF. Clearly $T(U) \subset (R \times R)^G$. Suppose $(c, d) \notin T(U)$. First, it will be shown that there is an x in U such that $c_x \neq d_x$. For $T' = \{(c,d): c_x = d_x \text{ for all } x \in X\}$ is an R-subalgebra of $R \times R$ which contains $(0, a)$ for all $a \in R$ such that $U \subset Z(a)$, so $T(U) \subset T'$. If $(c,d) \in T'$, $U \subset Z(d-c)$, and hence $(c,d) = (c,c) + (0, d-c)$ is in $T(U)$, so $T(U) = T'$. Now choose e such that $x \in N(e) \subset U$, which can be done since U is open. Let σ be the R-algebra automorphism of $R \times R$ which is the identity on $(R \times R)(1-e)$ and the transposition on $(R \times R)e$. Now if (a, b) is in $T(U)$ and $y \in N(e)$ then $a_y = b_y$ so $\sigma_y((a,b))_y = (a,b)_y$, while if $y \notin N(e)$, $\sigma_y = 1_y$ so $\sigma_y((a,b)_y) = (a,b)_y$ again, and hence $\sigma(a,b) = (a,b)$. Thus $\sigma \in G$. But $\sigma_x((c,d))_x = (d_x, c_x) \neq (c,d)_x$, so $\sigma(c,d) \neq (c,d)$. Thus $(c,d) \notin (R \times R)^G$, and the result follows.

LEMMA 20. *Let $a \in R$. Then $\{x \in X(R): T(Z(a))_x = R_x\} = Z(a)$.*

PROOF. If $x \in Z(a)$, then $(0,b)_x = 0$ if $Z(a) \subset Z(b)$, and hence $T(Z(a))_x = R_x$. Conversely, if $T(Z(a))_x = R_x$ then $(0,a)_x = (0,0)$ and hence $a_x = 0$ so $x \in Z(a)$.

The converse of Corollary 16 is now available.

PROPOSITION 21. *Suppose that for every strongly separable R-algebra S generated over R by idempotents and every group G of R-algebra automorphisms of S, the algebra S^G is locally strongly separable over R. Then R is zero-closed.*

PROOF. Let $a \in R$. By Lemma 19 and the hypotheses, $T(Z(a))$ is a locally strongly separable R-algebra. For x in $X(R)$, $T(Z(a))_x$ is a strongly separable subalgebra of $R_x \times R_x$, and hence is either R_x or $R_x \times R_x$. By [10, (2.11), p. 88], since $M = R \times R / T(Z(a))$ is a finitely generated R-module, $F = \{x \in X(R): M_x = 0\}$ is open. Thus $X(R) - F = \{x: T(Z(a))_x = R_x\}$ is closed, and by Lemma 20, this means $Z(a)$ is closed.

This leads, finally, to the main result.

THEOREM 22. *The ring of invariants of every group of R-algebra automorphisms of every strongly separable R-algebra is locally strongly separable if and only if R is extensionally zero-closed.*

PROOF. If R satisfies the first condition and R' is a strongly separable R-algebra, then Proposition 21 applied to $R' \times R'$ (which is a strongly separable R-algebra) shows that R' is zero-closed. The other half of the theorem is Proposition 15.

The theorem can be regarded as a characterization of extensionally zero-closed rings, while Proposition 21 and Corollary 16 give the corresponding characterization of zero-closed rings.

REFERENCES

1. S. Chase, D. Harrison and A. Rosenberg, *Galois theory and Galois cohomology of commutative rings*, Mem. Amer. Math. Soc. **52** (1965).

2. F. DeMeyer, *Separable polynomials over commutative rings*, Rocky Mt. J. Math. **2** (1972), 299–310.

3. L. Gillman and M. Jerison, *Rings of Continuous Functions*, D. Van Nostrand Co., Inc., Princeton, New Jersey, 1960.

4. A. Magid, *Locally Galois algebras*, Pacific J. Math. **33** (1970), 707–724.

5. A. Magid, *Galois groupoids*, J. Algebra **18** (1971), 89–102.

6. A. Magid, *The separable closure of some commutative rings*, Trans. Amer. Math. Soc. **170** (1972), 109–124.

7. R. Pierce, *Modules over commutative regular rings*, Mem. Amer. Math. Soc. **70** (1967).

8. O. Villamayor, *Separable algebras and Galois extensions*, Osaka J. Math. **4** (1967), 161–171.

9. O. Villamayor and D. Zelinsky, *Galois theory for rings with finitely many idempotents*, Nagoya Math. J. **27** (1966), 721–731.

10. O. Villamayor and D. Zelinsky, *Galois theory for rings with infinitely many idempotents*, Nagoya Math. J. **35** (1969), 83–98.

University of Oklahoma
Norman, Oklahoma, U. S. A.